

# Submanifolds in five-dimensional pseudo-Euclidean spaces and four-dimensional FRW universes

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## Abstract

Equations for submanifolds, which correspond to embeddings of the four-dimensional FRW universes in five-dimensional pseudo-Euclidean spaces, are presented in convenient form in general case. Several specific examples are considered.

It is well known that in general case a four-dimensional pseudo-Riemannian manifold can be represented as a submanifold in a flat ten-dimensional space-time [1]. If the metric possesses additional symmetries the dimensionality of the ambient space-time may be smaller, for example, it is well known that the de Sitter space  $dS_4$  can be represented as a hyperboloid in five-dimensional Minkowski space (for the first time explicit formulas for this embedding were presented in [2]). It is less known that the FRW (Friedmann-Robertson-Walker) space-times can also be embedded in five-dimensional Minkowski space (more generally, in pseudo-Euclidean spaces as we will show below). For the first time this fact was noticed almost 80 years ago in [3], where explicit formulas for the embeddings of the FRW universes were presented (see note D in [3]). In 1965 analogous formulas were presented in [4], where various embeddings of the solutions to equations of General Relativity in pseudo-Euclidean spaces were considered. Nevertheless, in spite of the recent activity in the field of embeddings of the four-dimensional General Relativity in five-dimensional flat or Ricci-flat space-times (see, for example, [5, 6] and references therein), it looks as if the results of [3, 4] on the embedding of the FRW space-times are not widely known (though they can be found in textbook [7], pp. 413–414), contrary to the case of the de Sitter space, which is discussed in many textbooks devoted to gravitation and cosmology (see, for example, [8, 9, 10]). Indeed, formulas for embeddings of the FRW space-times in five-dimensional Minkowski space were reinvented in [11] (for the open, closed and spatially-flat FRW universes) and in [12] (for the spatially-flat FRW universe). But although the explicit formulas of embeddings for the FRW universes in five-dimensional Minkowski space are known, we have failed to find equations for submanifolds, which correspond to such embeddings, in explicit and convenient form. In this note we will derive the known formulas for embeddings of the open and closed FRW universes (taking into account two possible embeddings for the open FRW universe) and obtain the corresponding equations for submanifolds. For completeness we will also present the equation for the case of the spatially-flat FRW universe. Finally, we will consider several specific examples.

First, we consider the case of the open universe. Let us take a five-dimensional space-time with the flat metric

$$ds^2 = -dt'^2 + d\bar{y}^2 \pm d\eta^2 = -dt'^2 + dr^2 + r^2 (d\theta^2 + \sin^2(\theta)d\varphi^2) \pm d\eta^2. \quad (1)$$

Note that the extra dimension with coordinate  $\eta$  can be space-like or time-like. With the help of transformations

$$r = \tau \sinh(\chi), \quad (2)$$

$$t' = \tau \cosh(\chi) \quad (3)$$

one can obtain the metric

$$ds^2 = -d\tau^2 + \tau^2 (d\chi^2 + \sinh^2(\chi) (d\theta^2 + \sin^2(\theta)d\varphi^2)) \pm d\eta^2, \quad (4)$$

which resembles the metric of the Milne universe [13] (for details see also [10], pp. 204–207; and "the expanding Minkowski universe" in [7], pp. 362–365 and in [9], pp. 743–744) apart from the term  $d\eta^2$ . Now it is easy to find the appropriate submanifold for the case of the open FRW universe. Let us suppose that the coordinates on the hypersurface are  $\chi, \theta, \varphi$  and  $t$ , such that

$$\tau = a(t), \quad \eta = b(t). \quad (5)$$

We get

$$ds^2 = -\left(\dot{a}^2(t) \mp \dot{b}^2(t)\right) dt^2 + a^2(t) (d\chi^2 + \sinh^2(\chi) (d\theta^2 + \sin^2(\theta)d\varphi^2)), \quad (6)$$

where  $\dot{\phantom{x}} = \frac{d}{dt}$ . Thus,

$$\text{if} \quad \dot{a}^2(t) > 1, \quad \dot{b}^2(t) = \dot{a}^2(t) - 1 \quad (\text{space-like extra dimension}), \quad (7)$$

$$\text{and if} \quad \dot{a}^2(t) < 1, \quad \dot{b}^2(t) = 1 - \dot{a}^2(t) \quad (\text{time-like extra dimension}), \quad (8)$$

we get the four-dimensional FRW metric of the open universe. Equations (2), (3), (5) with (7) can be found in [4, 11]. The case (8) was mentioned in [3] and considered in [7] as an embedding of the open universe. We will show below that it leads to some peculiar consequences and is rather unphysical in the standard FRW cosmology, contrary to the case (7).

The equation of the appropriate submanifold can be easily obtained. Indeed, using (2), (3) and (5) we get

$$t'^2 - r^2 = a^2(t) \Big|_{t=b^{-1}(\eta)}. \quad (9)$$

The latter formula can be rewritten in another form. Let us consider the Friedmann equation (here and below we consider the standard FRW cosmology)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}, \quad (10)$$

$\kappa = -1$  for the open universe. Using this equation one can obtain from (5), (7) and (8)

$$\eta = \pm \int dt \sqrt{\frac{8\pi G}{3}\rho a^2} = \pm \int da \sqrt{\frac{\frac{8\pi G}{3}\rho a^2}{1 + \frac{8\pi G}{3}\rho a^2}}, \quad (11)$$

( $\rho > 0$ , space-like extra dimension),

$$\eta = \pm \int dt \sqrt{\frac{8\pi G}{3}|\rho|a^2} = \pm \int da \sqrt{\frac{\frac{8\pi G}{3}|\rho|a^2}{1 - \frac{8\pi G}{3}|\rho|a^2}}, \quad (12)$$

( $\rho < 0$ , time-like extra dimension),

which leads to the following equations for the hypersurfaces:

$$\eta = \pm \int \sqrt{\left(\frac{\frac{8\pi G}{3}\rho a^2}{1 + \frac{8\pi G}{3}\rho a^2}\right) da} \Bigg|_{a=\sqrt{t'^2-r^2}}, \quad (13)$$

( $\rho > 0$ , space-like extra dimension),

$$\eta = \pm \int \sqrt{\left( \frac{\frac{8\pi G}{3} |\rho| a^2}{1 - \frac{8\pi G}{3} |\rho| a^2} \right) da} \Big|_{a=\sqrt{t'^2 - r^2}}, \quad (14)$$

( $\rho < 0$ , time-like extra dimension).

The energy density  $\rho = \rho(a)$  is defined by the corresponding equation(s) of state. Provided we know  $\rho(a)$ , we can easily get particular equation for the submanifold. We can also see that the choice (8) corresponds to the negative energy density of the matter (in this case  $\dot{a}^2 < 1$  as can be seen from (10)).

Now we turn to the case of the closed universe. Let us take a five-dimensional flat space-time with the metric

$$ds^2 = -dt'^2 + d\bar{y}^2 + d\eta^2 = -dt'^2 + dr^2 + r^2 (d\theta^2 + \sin^2(\theta) d\varphi^2) + d\eta^2 \quad (15)$$

and make a coordinate transformation

$$r = z \sin(\chi), \quad (16)$$

$$\eta = z \cos(\chi). \quad (17)$$

We obtain the metric

$$ds^2 = -dt'^2 + z^2 (d\chi^2 + \sin^2(\chi) (d\theta^2 + \sin^2(\theta) d\varphi^2)) + dz^2. \quad (18)$$

Now let us suppose that the coordinates on the hypersurface are  $\chi$ ,  $\theta$ ,  $\varphi$  and  $t$ , such that

$$z = a(t), \quad t' = b(t). \quad (19)$$

We get

$$ds^2 = -(\dot{b}^2(t) - \dot{a}^2(t)) dt^2 + a^2(t) (d\chi^2 + \sin^2(\chi) (d\theta^2 + \sin^2(\theta) d\varphi^2)). \quad (20)$$

Thus if

$$\dot{b}^2(t) = \dot{a}^2(t) + 1, \quad (21)$$

we get the four-dimensional FRW metric of the closed universe. We see that now the only possibility for the ambient space-time is to be (4+1), the case (3+2) being impossible. Equations (16), (17), (19) and (21) can be found in [3, 4, 7, 11].

The equation of the appropriate submanifold can also be easily obtained. Indeed, using (16), (17) and (19) we get

$$\eta^2 + r^2 = a^2(t) \Big|_{t=b^{-1}(t')}. \quad (22)$$

To rewrite it in a more useful form we will repeat the steps presented above for the open universe (but now using  $\kappa = 1$ ) and obtain

$$t' = \pm \int \sqrt{\left( \frac{\frac{8\pi G}{3} \rho a^2}{\frac{8\pi G}{3} \rho a^2 - 1} \right) da} \Big|_{a=\sqrt{\eta^2 + r^2}}. \quad (23)$$

For completeness we also present the formulas for embedding of the spatially-flat FRW universe in five-dimensional Minkowski space:

$$t' = \frac{1}{\alpha} \left( a(t)\vec{x}^2 + \int \frac{dt}{\dot{a}(t)} \right) + \alpha \frac{a(t)}{4}, \quad (24)$$

$$\eta = \gamma \left[ \frac{1}{\alpha} \left( a(t)\vec{x}^2 + \int \frac{dt}{\dot{a}(t)} \right) - \alpha \frac{a(t)}{4} \right], \quad (25)$$

$$\vec{y} = a(t)\vec{x}, \quad (26)$$

where  $\gamma = \pm 1$ ,  $\alpha$  is a constant with the dimension of length,  $\alpha \neq 0$ . Substituting (24)-(26) into (15) we easily obtain

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2, \quad (27)$$

which corresponds to a cosmology with zero spatial curvature. The parameter  $\alpha$  simply corresponds to the invariance of (27) under the rescaling  $a(t) \rightarrow a(t)\beta$ ,  $\vec{x} \rightarrow \vec{x}/\beta$ , where  $\beta \neq 0$  is a constant. Equations (24)-(26) can be found in [3, 7] for arbitrary  $\alpha$  and  $\gamma = -1$  (in our notations), in [4] with  $\alpha = 2$  and  $\gamma = 1$  and in [11] with  $\alpha = 2$  and  $\gamma = -1$ . They are also in agreement with the coordinate transformations presented in [14, 15] between the Ponce de Leon metric [16] and the five-dimensional Minkowski space (to see it one should set  $\alpha \rightarrow 2\alpha$ ,  $\gamma = -1$ ,  $a(t) = t^{\frac{1}{\alpha}}$  in (24)-(26) and  $l \equiv 1$  in eqs. (4a)-(4c) of [15]).

Equation for the submanifold in five-dimensional Minkowski space corresponding to the spatially-flat FRW universe has the form (it can be obtained from the one presented in note D of [3] (see also [12]) using the Friedmann equation (10) with  $\kappa = 0$ ):

$$t'^2 - \vec{y}^2 - \eta^2 = \frac{3}{8\pi G} \left[ a \int \frac{da}{a^2 \rho} \right]_{a=\frac{2(\gamma t' - \eta)}{\alpha}}. \quad (28)$$

Now let us turn to specific examples. Below we will present the explicit form of the appropriate hypersurfaces for the non-relativistic matter, the radiation and the cosmological constant, all with  $\rho > 0$ , i.e. for (4+1) ambient space-time. We will omit the integration constants coming from (13), (23) and (28), they can be eliminated by shifts of  $t'$  or  $\eta$ . We also set  $\gamma = 1$  in (28).

1. Matter dominated Universe,  $\rho = \rho_0 a^{-3}$ .

- Open universe.

$$\left( \frac{3}{16\pi G \rho_0} \eta \right)^2 - \frac{3}{8\pi G \rho_0} \sqrt{t'^2 - r^2} = 1.$$

- Closed universe.

$$\left( \frac{3}{16\pi G \rho_0} t' \right)^2 + \frac{3}{8\pi G \rho_0} \sqrt{\eta^2 + r^2} = 1.$$

- Spatially-flat universe.

$$t'^2 - r^2 - \eta^2 = C (t' - \eta)^3,$$

where  $C \neq 0$  is arbitrary constant of the corresponding dimensionality.

2. Radiation dominated Universe,  $\rho = \rho_0 a^{-4}$ .

- Open universe.

$$\sinh^2 \left( \sqrt{\frac{3}{8\pi G \rho_0}} \eta \right) = \frac{3}{8\pi G \rho_0} (t'^2 - r^2).$$

- Closed universe.

$$\sin^2 \left( \sqrt{\frac{3}{8\pi G \rho_0}} t' \right) = \frac{3}{8\pi G \rho_0} (\eta^2 + r^2).$$

- Spatially-flat universe.

$$t'^2 - r^2 - \eta^2 = C (t' - \eta)^4$$

with  $C \neq 0$ .

3. Cosmological constant,  $\rho = \rho_0 = \text{const}$ . For all three cases we get

$$t'^2 - r^2 - \eta^2 = -\frac{1}{H^2}$$

with  $H = \sqrt{\frac{8\pi G \rho_0}{3}}$ , which is of course the expected and the well-known result for  $dS_4$  space.

Note that since  $a(t)$  in (6) and (20) has the dimension of length, the dimension of  $\rho_0$  in the above formulas is different for different types of the matter.

Quite an interesting case is that of the cosmological constant with  $\rho = \rho_0 < 0$ . According to (14) we should take (3+2) ambient space-time. From (14) we get the equation of the hypersurface

$$t'^2 - r^2 + \eta^2 = \frac{3}{8\pi G |\rho_0|},$$

which is the well-known result for  $AdS_4$  space. We see that for the negative energy density of matter the FRW metric exists only for the case of the open universe and it can be represented as the metric induced on the submanifold in (3+2) pseudo-Euclidean space.

We would like to note that the simple examples presented above are not the only equations of submanifolds which can be obtained analytically. For example, for  $\rho(a) = \frac{\rho_m}{a^3} + \rho_\Lambda$  and  $\kappa = 0$ , which represents the main contribution to the energy density of the Universe at the present epoch, the equation of the corresponding submanifold can also be obtained analytically (but this equation has a rather bulky form and we do not present it here). As for more general cases, equations (13), (14), (23) and (28) allow one to obtain analytically or numerically (and visualize, as it was made in [11, 15, 17, 18, 19] for several cases) the corresponding surfaces in five-dimensional pseudo-Euclidean space using only equations of state of the matter (of course, if these equations of state suggest an explicit form of  $\rho(a)$  without solving the Friedmann equations). It is worth mentioning that such a visualization can be very informative, see essay [20] on this subject.

One more remark is in order. The Friedmann equation (10) corresponds to the standard four-dimensional FRW cosmology. In this sense the extra dimension in (1) is, of course, unphysical. Meanwhile, there are many multidimensional models which provide effective four-dimensional cosmology of the FRW type, though with dynamics different from the conventional one. For example, one can consider five-dimensional brane world models [21] or five-dimensional Ricci-flat models [5, 6], where extra dimension is physical and the five-dimensional space-time is not necessarily flat. In both cases the effective four-dimensional FRW metric induced on the brane

or on some hypersurface in this physical five-dimensional space-time again can be represented as the metric induced on the hypersurface in the unphysical five-dimensional pseudo-Euclidean space with the help of formulas (9), (22) and the corresponding equation for the spatially-flat case (of course, we can not use (13), (14), (23) and (28) because they were obtained with the help of the standard Friedmann equation (10), which is valid only in the standard FRW cosmology). In the general case these physical and unphysical five-dimensional space-times do not coincide. But there are some exceptions. As examples, one can consider the Ponce de Leon metric [16] and the five-dimensional analog of the de Sitter space [16] (the latter metric was also obtained in [22]), these solutions describe five-dimensional Ricci-flat space-times. It was noted in [23, 24] that both solutions are Riemann-flat space-times also. The explicit coordinate transformations between the five-dimensional Minkowski space and these solutions can be found in [14, 15] for the Ponce de Leon metric and in, for example, [25] for the five-dimensional analog of the de Sitter space. Thus, in these cases the physical five-dimensional space coincides with the unphysical one.

We hope that the result presented in this note can be useful, at least from historical and pedagogical points of view.

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